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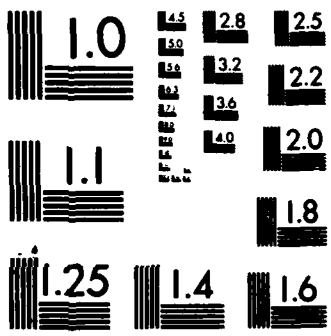

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THE DYNAMIC STEADY-STATE PROPAGATION OF AN ANTI-PLANE SHEAR CRACK IN A GENERAL LINEARLY VISCOELASTIC LAYER

JAY R. WALTON

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In a previous paper, the dynamic, steady-state propagation of an semi-infinite anti-plane shear crack was considered for an infinite, general linearly viscoelastic body. Under the assumptions that the shear modulus is a positive, non-increasing continuous and convex function of time, convenient, closed-form expressions were derived for the stress intensity factor and for the entire stress distribution ahead of and in the plane of the advancing crack. The solution was shown to have a simple, universal

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dependence upon the shear modulus and crack speed from which qualitative information can readily be gleaned. Here, the corresponding problem for a general, linearly viscoelastic layer is solved. An infinite series representation for the stress intensity factor is derived, each term of which can be calculated recursively in closed-form. As before, a simple universal dependence upon crack speed and material properties is exhibited.

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Problem Formulation

The analysis of dynamic fracture models, that is models based upon the equations of motion of continuum mechanics rather than the equations of equilibrium, has received considerable attention recently in the applied mechanics literature. Principally for the sake of mathematical convenience, most of these studies have been in the context of linear elasticity. Consequently, for elastic material, many canonical boundary value problems have been solved, either in closed form or numerically, for both steady-state and transient modes of crack propagation. For viscoelastic material, however, much less progress has been made in constructing convenient analytical solutions to even the simplest dynamic crack problems.

In [2], an analysis was presented of the dynamic, steady-state propagation of a semi-infinite, anti-plane shear crack in a general, infinite, homogeneous and isotropic linearly viscoelastic body. (See [2] for a discussion of other relevant studies of dynamic viscoelastic crack propagation.) With only very weak assumptions on the shear modulus (specifically, that it be a continuously differentiable, convex and non-increasing function of time), simple closed form expressions were constructed for the stress intensity factor and the entire stress field in front of and in the plane of the advancing crack. Moreover, it was shown that if v, c and c^* denote the crack speed and elastic shear wave speeds corresponding to the zero and infinite time values of the shear modulus, respectively, then for $0 < v < c^*$ the stress field is the same as for elastic material (that is, it is independent of crack speed and material properties), whereas for $c^* < v < c$ the stress field depends upon both v and material properties. An important result from this study was that the dependence upon v and material properties has a

simple universal form from which qualitative and quantitative information can be easily obtained.

In this paper, the corresponding problem for a viscoelastic layer of finite thickness is considered. The finite layer problem is of interest because the model has more relevance to engineering applications and significantly more mathematical complexity than does the infinite layer problem. Of additional importance is the fact that the Riemann-Hilbert method employed in [2] goes through for this more complicated problem. Specifically, the principal obstacle encountered in applying the Riemann-Hilbert method is the evaluation of a certain rather complx appearing combination of functions and integral transforms. It is demonstrated here that a simplification similar to that effected in [2] occurs also for the finite layer problem and that the stress intensity factor has a simple universal dependence on v, material properties and layer thickness, even for general shear modulus.

It should also be remarked that the approach adopted here has proved to be convenient for the calculation of the angular dependence of the near crack tip stress field for both the finite and infinite layer problems. These calculations are the subject of a forthcoming paper as is analysis of the substantially more complicated opening mode plane strain problem. This section concludes with the formulation of the appropriate boundary value problem and the derivation of the corresponding Riemann-Hilbert problem.

The governing field equations for the motion of the linearly viscoelastic layer are

$$\sigma_{ij,j} = \rho\, u_{i,tt} \;, \qquad -\infty < x_1 < \infty \;, \qquad -h \leq x_2 \leq h \;,$$

$$\varepsilon_{ij} = (u_{i,j} + u_{j,i})/2 \ ,$$

$$\sigma_{ij} = 2\mu{*}d\varepsilon_{ij} + \delta_{ij}\lambda{*}d\varepsilon_{kk}$$

where σ_{ij}, ε_{ij} and u_i denote the stress, strain and displacement fields, respectively. The summation convention is in effect, $f_{,i}$ denotes partial differentiation of the function f and $\mu{*}d\varepsilon$ denotes the Riemann-Stieltjes convolution

$$\mu{*}d\varepsilon = \int_{-\infty}^{t} \mu(t-\tau)d\varepsilon(\tau) \ .$$

Since the body is assumed to be in a state of anti-plane strain deformation, the only equation of motion not identically satisfied is

$$\mu{*}d\Delta u_3 = \rho \ u_{3,tt}$$

where Δu_3 denotes the 2-dimensional Laplacian, $\Delta = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2$. A semi-infinite crack is assumed to be propagating to the right with constant speed v along the x_1-axis. The crack is subjected to a traveling distribution of applied tractions, $\sigma_{23}(x_1,0,t) = f(x_1-vt)$ for $x_1 \leq vt$, while on the upper and lower surfaces of the layer two possible boundary conditions are considered, fixed grip and traction free, i.e.

$$\text{I.} \quad u_3(x_1,\pm h,t) = 0 \ , \quad -\infty < x_1 < \infty$$

$$\text{II.} \quad \sigma_{23}(x_1,\pm h,t) = 0 \ , \quad -\infty < x_1 < \infty \ .$$

Therefore, adoption of the Galilean variables $x = x_1 - vt$, $y = x_2$ yields the boundary value problem

$$\mu{*}d\Delta u_3 = \rho v^2 u_{3,xx} \qquad |y| < h, \ |x| < \infty \tag{1.1}$$

$$\sigma_{23}(x,0) = \frac{\partial}{\partial y}(\mu{*}du_3) = f(x) \ , \qquad x < 0$$

$$u_3(x,0) = 0 \ , \qquad x > 0 \tag{1.2}$$

$$\text{I.} \quad u_3(x,\pm h) = 0\ , \qquad |x| < \infty$$

or

$$\text{II.} \quad \sigma_{23}(x,\pm h) = 0\ , \qquad |x| < \infty\ . \tag{1.3}$$

The next series of steps are similar to those in [2] and only will be summarized here.

Use will be made of the Fourier transform defined by

$$\hat{f}(p,y) = \int_{-\infty}^{\infty} e^{ipx} f(x,y)\,dx = F_+(p,y) + F_-(p,y)$$

here

$$F_+ = \int_0^{\infty} e^{ipx} f(x,y)\,dx\ ,$$

$$F_- = \int_{-\infty} e^{ipx} f(x,y)\,dx\ .$$

Transforming 1.1, solving the resulting ordinary differential equation and applying the boundary conditions 1.3 yields

$$\hat{u}_3(p,y) = A_1(p)\begin{cases}\sinh(\gamma(p)(h-y)) & \text{I.}\\ \cosh(\gamma(p)(h-y)) & \text{II.}\end{cases}$$

with $\gamma^2 = p^2 + ipv\ /\ \hat{\mu}(-vp)$. As will be discussed later, it is convenient to choose a square root of γ^2 with positive real part. Applying the boundary conditions 1.2 produces the relation on $y = 0$

$$\hat{\sigma}^+_{23} + \hat{\sigma}^-_{23} = v\hat{\mu}(-vp)\gamma_1(p)\hat{u}_{3,1}\begin{cases}\coth\ (h\gamma(p))\ , & \text{I.}\\ \tanh\ (h\gamma(p))\ , & \text{II.}\end{cases} \tag{1.4}$$

where $\sigma^+_{23} + \sigma^-_{23}$ denote the restrictions of σ_{23} to the positive and negative x-axis, repsectively.

Equation 1.4 may be viewed as the Riemann-Hilbert problem

$$F^{+}(p) = G(p)F^{-}(p) + g(p) \qquad 1.5$$

where

$$F^{+}(p) = \hat{\sigma}_{23}^{+}$$

$$F^{-}(p) = \hat{u}_{3,1}^{-}$$

$$g(p) = -\hat{\sigma}_{23}^{-} = -\hat{f}$$

$$G(p) = v\hat{\mu}(-vp)\gamma(p)\begin{cases}\coth\ (h\gamma(p))\ , & \text{I.}\\ \tanh\ (h\gamma(p))\ , & \text{II.}\end{cases}\ .$$

It is easily demonstrated a posteriori that $\sigma_{23}^{+}(x,0)$ and $u_{3,1}^{-}(x,0)$ are such that $F^{+}(z) = \hat{\sigma}_{23}^{+}$ and $F^{-}(z) = \hat{u}_{3,1}^{-}$ define functions analytic for $I_m(z) \gtrless 0$, respectively, and are such that the limits

$$F^{\pm}(p) = \lim_{q\to 0^{\pm}} F^{\pm}(p+iq)$$

exist and satisfy 1.5. In the next section the Riemann-Hilbert problem 1.5 is solved and the stress intensity factor calculated.

2. Problem Solution

In order to solve 1.5 it is necessary to determine the mapping properties of the coefficient G(p). (See [1], for example, for a detailed discussion of the theory of Riemann-Hilbert boundary problems.) As in [2], it is convenient to rewrite G(p) as

$$G(p) = \text{sgn}(p)G_1(p)G_2(p)$$

where

$$G_1(p) = {=}i\hat{\mu}(-vp)\gamma_1(p) \qquad 2.1$$

$$G_2(p) = \begin{cases}\coth\ (h|p|\gamma_1(p))\ , & \text{I.}\\ \tanh\ (h|p|\gamma_1(p))\ , & \text{II.}\end{cases} \qquad 2.2$$

$$\gamma_1(p) = [1-\rho v^2/\bar{\mu}(-pv)]^{1/2} \qquad 2.3$$

$$\bar{\mu}(-vp) = ipv\hat{\mu}(-vp) = \mu(0) + \int_0^\infty e^{-ivpt}d\mu(t) \ . \qquad 2.4$$

For what follows it is sufficient (but clearly not necessary) to assume, as in [2], that the shear modulus, $\mu(t)$, is positive, continuously differentiable, non-increasing, convex and such that $\mu(\infty) = \lim_{t\to\infty} \mu(t) > 0$. These assumptions are still sufficiently general to include all physically reasonable examples. Moreover, an easy adaptation of the analysis presented here permits use of the pure power-law model, $\mu(t) = \mu_c(t/tc)^{-\alpha}$, which provides an effective characterization of many real materials.

For convenience the following observations from [2] are recorded here:

(i) $\bar{\mu}(0) = \mu(\infty) \leq \mathrm{Re}(\bar{\mu}(-vp)) \leq \mu(0) = \bar{\mu}(\infty)$

(ii) $\mathrm{Im}(\bar{\mu}(-vp)) = -\mathrm{Im}(\bar{\mu}(vp))$

(iii) $\arg(\bar{\mu}(-vp)) \begin{cases} \geq 0, & p > 0 \\ \leq 0, & p < 0 \end{cases}$

(iv) $\mathrm{Im}(\gamma_1^2(-p)) = \mathrm{Im}(\gamma_1^2(p))$

$\mathrm{Re}(\gamma_1^2(-p)) = \mathrm{Re}(\gamma_1^2(p))$

(v) $\mathrm{Im}(\gamma_1^2(p)) > 0, \quad 0 < p < \infty$

(vi) $1 - (v/c^*)^2 = \gamma_1^2(0) \leq \mathrm{Re}(\gamma_1^2(p)) \leq \gamma_1^2(\infty) = 1 - (v/c)^2$

where $c^* = (\mu(\infty)/\rho)^{1/2}$ and $c = (\mu(0)/\rho)^{1/2}$ are the elastic shear wave speeds corresponding to the value of $\mu(t)$ for infinite and zero time.

In order to take the square root of $\gamma_1^2(p)$ it is necessary to distinguish two cases

1. $0 < v < c^*$
2. $c^* < v < c$.

Taking the branch cut for $\gamma_1(p)$ to be the negative real axis assures that $\gamma_1(p)$ has positive real part. Hence for case 1., $\gamma_1(p)$ is Hölder continuous for all real p and

$$\text{(vii)} \quad \mathrm{Im}(\gamma_1(p)) = -\mathrm{Im}(\gamma_1(-p))$$

$$\mathrm{Re}(\gamma_1(p)) = \mathrm{Re}(\gamma_1(-p))$$

$$\text{(viii)} \quad \mathrm{Im}(\gamma_1(p)) > 0 , \quad 0 < p < \infty$$

$$\text{(ix)} \quad (1 - (v/c^*)^2)^{1/2} = \gamma_1(0) \le \gamma_1(\infty) = (1 - (v/c)^2)^{1/2} ,$$

whereas for case 2. (vii) and (viii) still hold but $\gamma_1(p)$ is discontinuous for $p = 0$. In particular,

$$\text{(x)} \quad \gamma_1(\pm\infty) = (1 - (v/c)^2)^{1/2}$$

$$\gamma_1(0\pm) = \pm i((v/c^*)^2 - 1)^{1/2} .$$

The image in the complex plane of the real p-axis under the transformation $\gamma_1(p)$ is illustrated in Fig. 1 of [2] for both cases 1. and 2.

Following the solution method employed in [2], we first consider the problem of finding functions $X^{\pm}(z)$ analytic for $\mathrm{Im}(z) \gtrless 0$, respectively, and which satisfy the homogeneous boundary relation

$$X^{+}(p) = G(p)\, X^{-}(p) . \qquad 2.5$$

Auxilliary functions $X_i^{\pm}(z)$, $i = 1,2$ are defined by

$$X_1^+(p) = G_i(p)\, X_i^-(-p) \ .$$

Then clearly,

$$X^{\pm}(z) = \omega^{\pm}(z)\, X_1^{\pm}(z)\, X_2^{\pm}(z) \qquad 2.6$$

where $\omega^{\pm}(z)$ denote branches of $z^{1/2}$ whose branch cuts are the negative and positive imaginary axes, respectively. (See [2].)

The functions $X_1^{\pm}(z)$ were constructed in [2], but here only the boundary limit $X_1^+(p)$ is needed. It was shown in [2] that

$$\omega^+(p)\, X_1^+(p) = \begin{cases} (-ip)^{1/2}|G_1(\infty)|^{1/2}\ , & \text{case 1.} \\ (q_o - ip)^{1/2}|G_1(\infty)|^{1/2}\ , & \text{case 2.} \end{cases}$$

where q_o is the unique positive solution to

$$vq_o \int_0^{\infty} \mu^*(t)e^{-q_o vt}dt = (v/c)^2 \ . \qquad 2.7$$

In 2.7 $\mu^*(t)$ denotes a normalized modulus given by

$$\mu^*(t) = \mu(t)/\mu(0) \ .$$

The central focus of this paper is the construction of the asymptotic form of $\sigma_{23}^+(x,0)$ for x near zero. To that end, much of the derivation presented in [2] is valid here also. In particular, it can be concluded that

$$F^{\pm}(z) = X^{\pm}(z)\frac{1}{2\pi i}\int_{-\infty}^{\infty} g(\tau)/X^+(\tau)\frac{d\tau}{\tau - z} \quad ,$$

$$\sigma_{23}^+(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ixp}\, X^+(p)dp\, \frac{1}{2\pi i}\int_{-\infty}^{\infty} g(\tau)/X^+(\tau)\frac{d\tau}{\tau - p}$$

and for x near zero, the dominant term in the asymptotic expansion of $\sigma_{23}^+(x)$ is given by

$$\sigma_{23}^{+}(x) \sim x^{-1/2} \frac{|G_1(\infty)|^{1/2}}{\sqrt{\pi}} \frac{1}{2\pi} \int_{-\infty}^{\infty} g(\tau)/X^{+}(\tau)d\tau = K\, x^{-1/2} . \qquad 2.8$$

The coefficient K is the stress intensity factor. It should be remarked that 2.8 is valid for the finite layer problem because $G_2(\infty) = 1$.

In [2] a simple closed form expression for K was constructed through the device of introducing the function h(x) for which

$$1/X^{+}(\tau) = \int_{-\infty}^{\infty} e^{-i\tau x} h(x)dx \qquad 2.9$$

and from which it follows that

$$K = -|G_1(\infty)|^{1/2} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \sigma_{23}^{-}(x)h(x)dx . \qquad 2.10$$

The principal result of [2] was the proof that

$$h(x) = |G_1(\infty)|^{-1/2} H(-x)|x|^{-1/2} \begin{cases} 1, & \text{case 1.} \\ e^{xq_o} , & \text{case 2.} \end{cases} \qquad 2.11$$

where H(x) is the Heaviside step function,

$$H(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0 . \end{cases}$$

For a layer of finite thickness, lines 2.8-2.10 are still valid, but with $X^{+}(\tau)$ given by 2.6. Consequently, the simple form 2.11 for h(x) does not hold; the contribution from $X_2^{+}(x)$ must be incorporated into the solution. It is convenient to remark here that since $G_2(p)$ for problems I and II are just reciprocals, the same is true for $X_2^{\pm}(z)$. Therefore, it is necessary to consider only one of the cases, say I.

Evaluating $X_2^{+}(p)$ is the central problem. Clearly, from the general theory of Riemann-Hilbert boundary value problems ([1]), it follows that

$$X_2^{\pm}(z) = e^{\Gamma_2^{\pm}(z)}$$

with

$$\Gamma_2^{\pm}(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \log\ (G_2(\tau))\frac{d\tau}{\tau - z}\,. \tag{2.12}$$

The integral in 2.12 can be evaluated by a technique similar to that employed in the analysis in [2] of the case $c^* < v < c$. Moreover, only the boundary limit $X_2^{+}(p)$ is needed in the computation of the stress field. As in [2], it is necessary to consider the two cases 1. and 2. separately.

Case 1. $\quad 0 < v < c^*$

The function $\log\ (G_2(\tau))$, $\tau > 0$ has a natural extension, $\log\ (G_2(z))$, to a function analytic in the fourth quadrant, i.e. for $z = \tau - iq$ with $\tau, q > 0$. Computing the boundary limit $\lim_{\tau \to 0+} \log\ (G_2(\tau - iq))$ yields

$$\begin{aligned}\lim_{\tau \to 0+} \log\ (G_2(\tau - iq)) &= \log\ (\coth\ (-iqh\gamma_1(-iq))) \\ &= \log\ (i\ \cot(hq\gamma_1(-iq)))\,.\end{aligned} \tag{2.13}$$

Since $q\gamma_1(-iq)$ is an increasing function of q, there exists a sequence of numbers, $\{a_n\}_{n=0}^{\infty}$, such that

$$\begin{aligned} &ha_{2n}\gamma_1(-ia_{2n}) = n\pi\ , \qquad n = 0,1,\ldots \\ &ha_{2n+1}\gamma_1(-ia_{2n+1}) = (n+1/2)\pi\ , \qquad n = 0,1\ldots\ . \end{aligned} \tag{2.14}$$

In light of 2.3, 2.14 can be rewritten

$$\begin{aligned} &va_{2n}\int_0^{\infty}\mu^*(t)\exp[-vta_{2n}]dt = (v/c)^2/[1-(n\pi/ha_{2n})^2] \\ &va_{2n+1}\int_0^{\infty}\mu^*(t)\exp[-vta_{2n+1}]dt = (v/c)^2/[1-\ (n+1/2)\pi/ha_{2n+1})^2]\ . \end{aligned} \tag{2.15}$$

Consequently,

$$\arg\,(i\cot\,(hq\gamma_1(-iq))) = \pi/2 \quad \text{for} \quad a_{2n} < q < a_{2n+1}\,,$$

while

$$\arg\,(i\cot\,(hq\gamma_1(-iq))) = -\pi/2 \quad \text{for} \quad a_{2n+1} < q < a_{2(n+1)}\,.$$

If $\tau < 0$, the function $\log\,(G_2(\tau))$ has the natural analytic extension $\log\,(G_2(z))$, $z = \tau - iq$, $q > 0$ given by $\log\,(G_2(z)) = \log\,(\coth(-hz\,\gamma_1(z)))$. Letting z approach the negative imaginary axis results in

$$\begin{aligned}\lim_{\tau\to 0-}\ \log\,(G_2(\tau-iq)) &= \log\,(\coth\,(ihq\gamma_1(-iq)))\\ &= \log\,(-i\cot\,(hq\gamma_1(-iq)))\,. \end{aligned} \qquad 2.16$$

The function $\Gamma_2^+(z)$ may be computed by replacing the integral in 2.12 over the real axis with the appropriate integrals along the lines $\tau - ia_{2n+1}$, $-\infty < \tau < \infty$ and $-iq$, $0 < q < a_{2n+1}$. Specifically, for $\mathrm{Im}(z) > 0$,

$$\Gamma_2^+(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \log\,(G_2(\tau))\frac{d\tau}{\tau - z}$$

$$= \frac{1}{2\pi i}\left[\int_{\infty - ia_{2n+1}}^{(0+)-ia_{2n+1}} + \int_{(0+)-ia_{2n+1}}^{(0+)-io} + \int_0^{\infty} \log(G_2(\tau))\frac{d\tau}{\tau - z}\right] \qquad 2.17$$

$$= \frac{1}{2\pi i}\left[\int_{-\infty}^{o} + \int_{(0-)-io}^{(0-)-ia_{2n+1}} + \int_{(0-)-ia_{2n+1}}^{-\infty - ia_{2n+1}} \log(G_2(\tau))\,\frac{d\tau}{\tau - z}\right] \qquad 2.18$$

$$= \frac{1}{2\pi i}\left[\int_{(0+)-io}^{(0+)-ia_{2n+1}} + \int_{(0-)-ia_{2n+1}}^{(0-)-io} \log(G_2(\tau))\frac{d\tau}{\tau - z}\right] \qquad 2.19$$

$$+\frac{1}{2\pi i}\int_{-\infty - ia_{2n+1}}^{\infty - ia_{2n+1}} \log(G_2(\tau))\frac{d\tau}{\tau - z}\,. \tag{2.20}$$

Lines 2.17 and 2.18 vanish by Cauchy's theorem. The integral in line 2.20 tends to zero as $n \to 0$ uniformly in z for $\mathrm{Im}(z) > 0$ and therefore will be denoted by $\varepsilon(n)$.

From the above observations and after an obvious change of variables in 2.19 it is now clear that

$$\Gamma_2^+(z) = \varepsilon(n) + \frac{1}{2\pi i}\int_0^{a_{2n+1}}[\log(i\cot(\mathrm{th}\gamma_1(-it))) - \log(-i\cot(\mathrm{th}\gamma_1(-it)))]\frac{dt}{t-iz}$$

$$= \varepsilon(n) + \frac{1}{2}\int_0^{a_1}\frac{dt}{t-iz} + \frac{1}{\pi}\sum_{j=1}^{n}\int_{a_{2j-1}}^{a_{2j+1}} \arg\,(i\cot(\mathrm{th}\gamma_1(-it)))\frac{dt}{t-iz}$$

$$= \varepsilon(n) + \frac{1}{2}\log\left(\frac{a_1 - iz}{-iz}\right) - \frac{1}{2}\sum_{j=1}^{n}\int_{a_{2j-1}}^{a_{2j}} - \int_{a_{2j}}^{a_{2j+1}}\frac{dt}{t-iz}$$

$$= \varepsilon(n) + \frac{1}{2}\log\left(\frac{a_1 - iz}{-iz}\right) - \sum_{j=1}^{n}[\log(a_{2j}-iz) - \log(a_{2j-1}-iz) - \log(a_{2j+1}-iz)]. \tag{2.21}$$

In 2.21, the limit $\mathrm{Im}(z) \to 0+$ is easily taken from which it then follows that

$$X_2^+ = (ip)^{-1/2}\, Y_n(p)\, e^{\varepsilon(n)} \tag{2.22}$$

with

$$\bar{X}_n(p) = \prod_{j=1}^{n} \left[\frac{a_{2j-1} - ip}{a_{2j} - ip}\right](a_{2n+1} - ip)^{1/2} . \qquad 2.23$$

Case 2. $c^* < v < c$

In contrast to case 1., for case 2. $\gamma_1(\tau)$ has a natural extension, $\gamma_1(z)$, that is analytic for $Im(z) < 0$ except for a branch cut, $-iq$, $0 < q < q_o$ with q_o given by 2.7. The same is then true for $\log(G_2(\tau))$ and in particular, $\log(G_2(z))$ is analytic for $Im(z) < -q_o$. After this minor modification is taken into account, the argument utilized in case 1. applies also for case 2. and yields the result

$$X_2^+(p) = (q_o - ip)^{-1/2} X_n(p) e^{\varepsilon(n)} \qquad 2.24$$

where $X_n(p)$ is still given by 2.23 and the a_n by 2.14. It should be observed that for case 2, the a_n are such that $q_o < a_1 < \ldots$ whereas for case 1, $0 < a_1 < \ldots$. Moreover, the a_n for case 2 are larger than the corresponding a_n for case 1.

Calculating $\sigma_{23}^+(x,0)$ requires the function $\bar{X}^+(p)$. From 2.6, 2.8-2.10, 2.22-2.24 it is easily seen that for both case 1 and case 2

$$X^+(p) = \lim_{n\to\infty} |G_1(\infty)|^{1/2} \prod_{j=1}^{n} \left(\frac{a_{2j-1} - ip}{a_{2j} - ip}\right)(a_{2n+1} - ip)^{1/2} , \qquad 2.25$$

$$= \lim_{n\to\infty} X_n^+(p)$$

and that

$$K = \lim_{n\to\infty} K_n , \quad K_n = -|G_1(\infty)|^{1/2} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{0} \sigma_{23}^-(x,0) h_n(x) dx$$

with

$$h_n(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ixp}/\underline{\Sigma}_n^{+}(p)\,dp. \qquad 2.26$$

A convenient scheme for inductively approximating the stress intensity factor K can be based upon the observation that

$$\prod_{j=1}^{n}\left(\frac{a_{2j}-ip}{a_{2j-1}-ip}\right) = 1 + \frac{b_{n1}}{(a_1-ip)} + \ldots + \frac{b_{nn}}{(a_{2n-1}-ip)}\,, \qquad 2.27$$

where

$$\begin{aligned} b_{nk} &= \left[\prod_{j=1}^{n}(a_{2j} - a_{2k-1})\right]/\left[\prod_{j\neq k}^{n}(a_{2j-1} - a_{2k-1})\right] \\ &= b_{(n-1)k}\left(\frac{a_{2n}-a_{2k-1}}{a_{2n-1}-a_{2k-1}}\right). \end{aligned}$$

In light of 2.25-2.27 and after some routine integrations it can be shown that for $x < 0$,

$$h_n(x) = |x|^{-1/2}k_n(x)$$

$$k_n(x) = e^{a_{2n-1}x} - 2x\sum_{j=1}^{n} b_{nj}\int_0^1 \exp[x(a_{2n+1}t^2 + a_{2j-1}(1-t^2))]dt\,.$$

References

1. F.D. Gakov, Boundary Value Problems, Pergamon, London, 1966.

2. J.R. Walton, "On the Steady-State Propagation of an Anti-Plane Shear Crack in an Infinite General Linearly Viscoelastic Body," Quart. Appl. Math., April 1982, 37-52.

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